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THE MIXING EFFECT OF SURFACE WAVES

BY CHAS. S. SLICHTER

THE following results are based upon the assumption that the "mixing effect" when a portion of a liquid is changed in shape is proportional to the magnitude of the shear. If we confine our attention to motion in two dimensions only, the problem before us is to determine to what extent particles of a liquid migrate and take up positions adjacent to particles of liquid not formerly their neighbors when a prism of liquid of square cross section changes to a prism of rectangular cross section. Theory requires that the particles of a "perfect" liquid hold the same relative position to each other after change as before. The extent to which this is not true in the case of an actual liquid constitutes what is designated as "mixing." Thus if the temperature of a cube of liquid varies uniformly from one of the faces to the opposite face, then after shear the relative distribution of temperature should be precisely the same as before shear. The liquid is "mixed" to the extent that the relative distribution of temperature has been disturbed.

If the dimensions of a rectangular parallelopiped of a liquid change from 1, 1, 1, to $1 + a$, 1, $1/(1 + a)$, the liquid is said to have experienced a finite shear of amount a. Hence, according to this definition, unit finite shear corresponds to change of cube 1, 1, 1, to parallelopiped 2, 1, 1/2. For rates of movement sufficiently low so that turbulent motion does not result, the initial assumption made above that mixing is proportional to shear seems sufficiently obvious if the present definition* of the magnitude of finite shear be adopted.

It is obvious that the amount of mixing that takes place in an element of liquid when subjected to unit shear must depend upon properties of the liquid which can be readily determined only by experiment. The viscosity, the density, and various molecular properties probably all play important parts in determining the amount of mixing that takes place when an element of a liquid changes shape.

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* It should be noted that this is not the definition adopted in the usual treatment of the subject of finite shear.

The primary cause of mixing in liquids, as ordinarily observed, is due to the turbulent motion that is set up when the differential velocity from layer to layer, or between the liquid and the boundary reaches a certain critical amount. It can readily be seen by consulting the diagrams in the papers by Hele-Shaw in *Nature*, Vol. 50, page 34, and Vol. 60, page 446, that slow movements in which turbulent motions have been eliminated result in a very small amount of mixing, although the amount of shear be relatively large. In the wave motion of a liquid on whose surface simple progressive waves are maintained, the deformation or strain of any element is periodic and the effect of mixing must be cumulative even though the motion be so slow that turbulent motion is not set up. The mixing effect of the waves can be expressed as a function of the various constants that particularize the special wave motion under consideration. It is obvious that the variation of the mixing effect with the depth of the particle of liquid below the surface is the first problem to be considered.

The following notation is used in what follows:

Height of trochoidal wave	H	in	meters
Wave length	λ	"	"
Depth of water	d	"	"
Distance from the bottom to particle under consideration . .	y	"	"

The well known theory of trochoidal surface waves enables us to write down the following results:

The radius of the circular orbit of a particle at any depth is:

$$r = m\ H/2 = \tfrac{1}{2} He^{-2\pi(y-d)\lambda}, \tag{1}$$

in which m is a proper fraction, the ratio of the radius of the orbit at level y, to the radius of the orbit of a surface particle.

The period of the wave motion in seconds is, if λ be measured in meters:

$$T \doteq \sqrt{2\pi\lambda/g} \doteq \frac{4}{5}\lambda^{\frac{1}{2}}; \qquad d \gtreqless \lambda, \tag{2}$$

the formula being correct to about the sixth decimal place. The error remains less than 5 per cent for all values of d satisfying*

$$4d > \lambda < d$$

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* The approximation consists in omitting $\tanh 2\pi d/\lambda$ from the denominator of the theoretical expression for T^2.

Consider two particles of water which lie in the same vertical line when the crest and also when the trough of the wave passes immediately above them. Let the radius of the circular orbit of the higher particle be r_2 and of the lower particle be r_1 and let the distance between the centers of the two circular orbits be a, which may be taken as small as we please. Then the distances between these particles are,

at crest, $a + r_2 - r_1$;
at trough, $a - r_2 + r_1$.

Therefore the small rectangular parallelopiped at the crest, of edges

$$1 + (r_2 - r_1)/a, \qquad 1, \qquad 1/[1 + (r_2 - r_1)/a],$$

is first sheared to the cube

$$1, \qquad 1, \qquad 1,$$

at an intermediate stage, and then this cube is sheared at the trough to the rectangular parallelopiped

$$1 - (r_2 - r_1)/a, \qquad 1, \qquad 1/[1 + (r_2 - r_1)/a].$$

The total finite shear from crest to trough is therefore of magnitude:

$$2(r_2 - r_1)/a,$$

or, in the limit,

$$2dr/dy.$$

Therefore the "mixing effect" per second of time, or the coefficient of mixing, is

$$E = 4Adr/dy \div \frac{4}{5}\lambda^{\frac{1}{2}} = 15.6\ AH\,\lambda^{-\frac{3}{2}}e^{2\pi(y-d)/\lambda}, \qquad (3)$$

in which A is the mixing effect, or specific coefficient of mixing, due to unit shear. This factor must be experimentally determined for each liquid.

Assume waves of length 10 meters and of height one meter in water whose undisturbed depth is 10 meters, then the rate of mixing at level y is given by

$$E = E_0\, e^{2\pi(y-10)/10}, \qquad (4)$$

in which E_0 is the rate of mixing at the surface.

Table I shows the coefficient of mixing, or the mixing effect for one second of time, of a trochoidal progressive wave one meter high and of various lengths. The results are expressed in terms of the mixing effect due to unit finite shear.

TABLE I

Depth in meters below surface	Length of waves in meters.						
	5	10	15	20	25	30	40
0	1.310	0.495	0.296	0.175	0.125	0.125	0.062
1	0.406	0.264	0.176	0.127	0.099	0.077	0.053
2	0.121	0.138	0.115	0.092	0.077	0.062	0.045
3	0.032	0.086	0.076	0.068	0.059	0.050	0.040
4	0.009	0.048	0.048	0.049	0.046	0.040	0.034

The above results give the measure of the migration of particles away from their initial location or original companions. The numbers may be conveniently designated, as above, "coefficients of mixing," although mixing, in the common meaning of the term, does not take place unless another condition be present, namely, unless there be a quality varying in magnitude throughout the liquid which can be modified by the migrating particles. We do not speak of "mixing" a liquid by stirring it unless it has a different quality in some parts than in others. Migrating particles can produce no change in a perfectly homogeneous liquid. For example, if a liquid be of homogeneous temperature throughout, it will remain of homogeneous temperature no matter how turbulent the motion. If, however, the initial temperature be not homogeneous (say it falls continuously from the surface downward, as it actually does in summer in the water of a lake or pond) then the migration of particles discussed above will bring about a new distribution of temperature. The new distribution depends as much upon the gradients in the initial distribution as it does upon the coefficient of mixing. This problem will be considered only in its special bearing upon changes of temperature taking place in a sheet of water due to progressive waves upon its surface. We do not aim to consider the mixing due to the turbulent waves near shore, nor to the effects of the wind in driving the warm surface water to lee shores. Professor E. A. Birge, who has given wide attention to all these matters, fully discusses the subject from the physical and practical side in another place.*

Let there be any quality or characteristic that remains constant in any horizontal layer of the liquid but varies in the vertical direction. For example, let there be a vertical distribution of temperature either continuous or discontinuous. If the temperature decrease with the depth below the surface there will be no tendency to set up convection currents, as long as the temperature is above 4°. When surface waves are set up in the liquid, the quality or characteristic pertaining to the various horizontal layers will gradually change, the distribution becoming immediately continuous across the former surfaces of discontinuity. Let the initial vertical distribution be defined by the equation,

$$t = f(y) \tag{5}$$

in which y is the depth of any point below the surface of the liquid [the same as $(y - d)$ above] and in which f represents a single valued function of y. Let the coefficient of mixing at any level be $\phi(y)$. At this level the initial gradient of t is

$$dt/dy = f'(y),$$

and after mixing it is

$$dt/dy = [1 - \phi(y)]f'(y). \tag{6}$$

Whence the distribution of the quality t after the "first mixing", or first unit of time, is given by

$$t_1 = f(y) - \int \phi(y)\, f'(y)\, dy + c.$$

Likewise the distribution of t after the nth interval is given by

$$t_n = \int [1 - \phi(y)]^n f'(y)\, dy + c. \tag{7}$$

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* The results of these extensive studies of the biological, chemical, and physical phenomena of fresh water lakes is appearing as a series of Monographs published by the Wisconsin Geological Survey. The chemical phenomena are treated in Bulletin XXII. The physical studies will appear in a bulletin now in preparation.

In the important special case in which the mixing is due to waves, we may write

$$\phi(y) = ae^{-by}. \tag{8}$$

The forms of a and b have been found above. In like manner

$$t_n = \int (1 - a e^{-by})^n f'(y)\, dy + c, \tag{9}$$

which permits of simplification for special values of f.

Consider the case in which the temperature falls from the surface to the bottom as a linear function of y, so that $f(y) = -ky$. Then

$$t_n = -k \int (1 - a e^{-by})^n dy + c, \tag{10}$$

which, for large values of n, becomes

$$t_n = -k \int e^{-nae^{-by}}\, dy = -\frac{k}{b} \operatorname{li} e^{-nae^{-by}}, \tag{11}$$

in which li stands for the Integral Logarithmus.*

The constant of integration must be determined by a quadrature. Mechanical quadrature was used in obtaining the numerical results given below.

The important case leading to the development of $f(y)$ in a Fourier's Series may also be written down.

Let

$$f(y) = k_p \sin py,$$

then, for large values of n

$$t_n = k_p \sum_{\alpha=0}^{\alpha=\infty} (-na)^\alpha \, (a^2b^2 + p^2)^{-\frac{1}{2}} \, (1/\alpha!) \sin(py - \theta), \tag{12}$$

in which $\theta = \tan^{-1}(ab/p)$.

Returning to the case of equation (11), the Integral Logarithmus permits numerical results to be obtained as follows: Let $H = 1$ meter, $\lambda = 10$

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* The Integral Logarithmus is defined by the integral $\int_0^x du/\log u$. See N. Nielsen, *Theorie des Integrallogarithmus und verwandter Transzendenten*, Leipzig, 1906. Since the publication (Leipzig, 1909) of Jahnke and Emde's *Funktionentafeln*, tables of this and the allied functions have become readily accessible. Table II was checked by use of the table of $Ei(-x)$ on page 21 of Jahnke and Emde's tables.

meters, $d = 10$ meters, and let n be sufficiently large so that appreciable values may be assigned to nA. From previous notation,

$$a = 15.6\ A/\lambda^{3/2} = 0.484A;$$

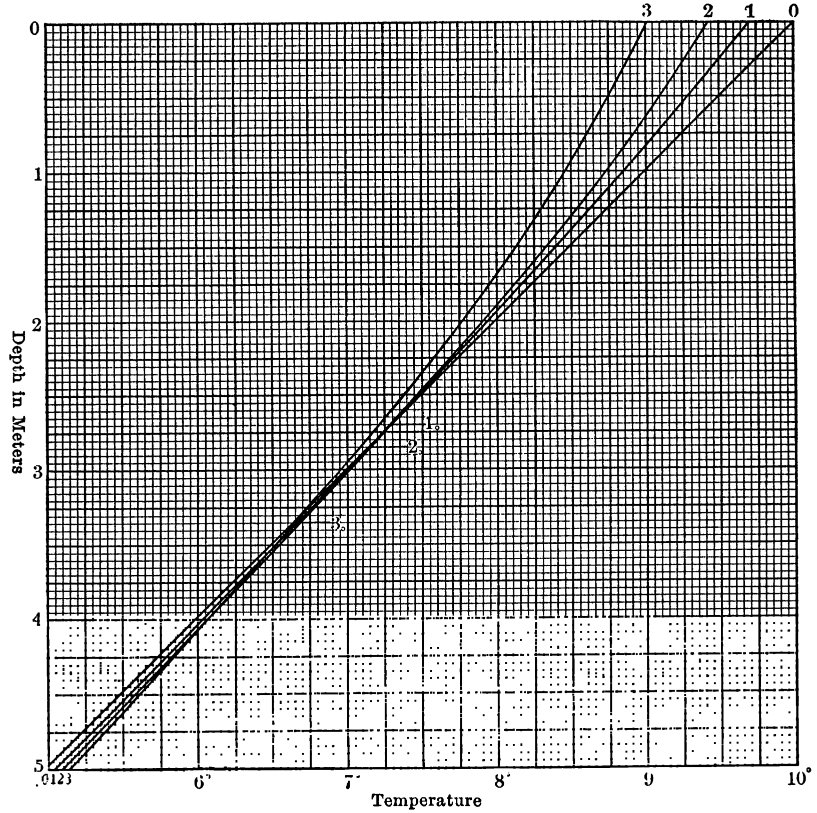
$$b = 2\pi/\lambda = 0.6283; \qquad e^{-by} = (0.534)^y.$$

As the value of A is not known, the amount of mixing after a given number of seconds cannot be found in absolute units but the amount of mixing at given *relative times* can readily be determined, since nA appears as a single variable in the results. With the above numerical data, and with nA taken equal to 1/2, 1, 2, or to elapsed times proportional to 1, 2, 4, the table II has been prepared.

Table II gives the variation in temperature in a sheet of water 10 meter deep, due to the mixing effect of surface waves of wave length 10 meters and height 1 meter, the initial temperature being equal to $(10 - y)$, where y is the depth below the surface of the water.

TABLE II

Elapsed time, nA =	0	½	1	2
Value of constant c =	0	0.069	0.13	0.215
Depth, y, below surface	Temp. degrees	Temp. degrees	Temp. degrees	Temp. degrees
0	10	9.70	9.45	9.06
1	9	8.87	8.75	8.48
2	8	7.96	7.92	7.78
3	7	7.01	7.01	6.98
4	6	6.04	6.07	6.09
5	5	5.05	5.11	5.15
6	4	4.06	4.11	4.18
7	3	3.06	3.12	3.19
8	2	2.07	2.12	2.20
9	1	1.07	1.12	1.21
10	0	0.07	0.13	0.22



The results are best shown graphically as in the figure. Here the original temperature is represented by the straight line 0 0. The modified temperatures at time 1/2 , 1 , 2 , 4 , are shown by the curved lines. The diagram is drawn for the upper 5 meters only. The curve for the fourth period is sketched only approximately, as the computation difficulties did not warrant greater exactness. It will be seen that the effect of the waves is gradually to change the temperature of the surface layers to a nearly uniform temperature, the absolute changes in the intervals of time being less than in earlier equal intervals. The changes in temperature show a tendency to cease rather abruptly at points where the curves cross the original gradient, as at 1_0, 2_0, 3_0. These are at depths of about 3 meters below the surface for the wave length and the wave height under consideration and gradually move downward with the lapse of time.

The same diagram may be used to show approximately the mixing effect due to waves of twice the height but of same wave length. In this case the curve 22 for wave height 2 meters corresponds to the same interval of time as the curve 11 for wave height 1 meter. The points 1, 2, 3 move to the left and the points 1_0, 2_0, 3_0, move downward at twice the rate in the case of the higher waves than in the case of the lower waves.

Although the mixing effect in the slow motion of a liquid is a magnitude of low order, yet there are important cases of motion in which the migrations of particles across the lines of flow is very great. An important case occurs in the motion of groundwater. It can be shown in this case that the movement of particles across the lines of flow equals about 30 per cent of the onward movement in the line of resultant motion. This means, for example, that impurities or contaminations are expanded or spread out over wide areas during the motion of underground waters. Diagrams showing some experimental manifestation of this fact will be found in Water Supply Paper No. 140 of the U. S. Geological Survey, pp. 22–68. The consideration of the mixing, or spread of contaminations, in groundwaters lead to the statement of a very general problem of much interest, which the writer will consider in another place.

The magnitude of the coefficient A assumed above cannot be even approximately assigned with the experimental data now at hand. It seems reasonably certain, however, that its magnitude will range higher than even the largest of the coefficients of diffusion.

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